СЕМЕЙСТВА ТЕОРИЙ И ИХ ХАРАКТЕРИСТИКИ

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Theories, Models, and Types

- (Elementary) Theory T: an information written by first-order formulas / **syntax**
- Complete Theory: a maximal consistent information
- Model $\mathcal{M}$ of T: an object realizing T / **semantic**
- Countable Model (Structure): a model (of a theory) with countably many elements
- (Complete) type: a (complete) information about a finite set A in $\mathcal{M}$

Spectrum functions

$I(T,\lambda)$ denotes the number of pairwise non-isomorphic models of T and having λ elements (λ is an infinite power, T is a complete theory without finite models).

The *spectrum function* $I(T,\cdot)$ maps a (finite or infinite) power $I(T,\lambda)$ for an infinite power λ.

We consider $\lambda=\omega$ (countable, i.e., enumerable by natural numbers forming the set ω).

It is known that for any countable complete theory T,

- $I(T,\omega)\neq 2$, **Vaught Theorem**,
- if $I(T,\omega)>\omega_1$ then $I(T,\omega)=2^\omega$, **Morley Theorem**.

Thus, $I(T,\omega)\in(\omega\setminus\{0,2\})\cup\{\omega,\omega_1,2^\omega\}$.

Countable theories with respect to countable models

Countable theories

ω-categorical

$I(T,\omega) = 1$

with finitely many countable models, Ehrenfeucht

$3 \leq I(T,\omega) < \omega$

with countably many countable models

$I(T,\omega) = \omega$

with ω_1 countable models

$I(T,\omega) = \omega_1$

Vaught Problem

with continuum many countable models

$I(T,\omega) = 2^\omega$

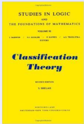
Theories

Greece, Athens, Acropolis, Theorias street

Models

Models

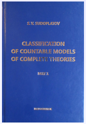
uncountable

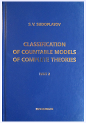
Shelah's 

,
B.Hart, E.Hrushovski,
M. S. Laskowski,
and many other specialists

countable

Many results
by specialists
including the author's





Countable models

Countable models

- **prime** over finite sets, almost prime (finitely generated)
- **limit** (not prime but unions of almost prime models)
- **other** (neither prime nor limit)

Countable models and types

We denote by

- $P(T)$ the number of pairwise non-isomorphic (almost) prime models of T,
- $L(T)$ the number of pairwise non-isomorphic limit models of T,
- $\mathrm{NPL}(T)$ the number of pairwise non-isomorphic other countable models of T.

The set of all types of theory T is denoted by $S(T)$.

Countable theories and types

Countable theories

small,
i.e., with
countably
many types

$|S(T)| = \omega$

unsmall,
i.e., with
continuum
many types

$|S(T)| = 2^{\omega}$

Countable models of small theories

Small theories

$\Rightarrow \mathrm{NPL}(T) = 0$

$I(T,\omega) = 1$

$P(T) = 1$
$L(T) = 0$

$3 \leq I(T,\omega) < \omega$

$1 < P(T) < \omega$
$1 \leq L(T) < \omega$

$I(T,\omega) = \omega$

$1 < P(T) \leq \omega$
$1 \leq L(T) \leq \omega$
$P(T) + L(T) = \omega$

$I(T,\omega) = \omega_1$

$1 < P(T) \leq \omega$
$L(T) = \omega_1$,
existence of T is unknown

$I(T,\omega) = 2^\omega$

$1 < P(T) \leq \omega$
$L(T) = 2^\omega$

Triples for distributions of countable models of theories with continuum many types[1]

THEOREM (R.A. Popkov, S.V. Sudoplatov, 2015)

Assuming the continuum hypothesis, for any theory T in the class $\mathcal{T}_c$ of theories with continuum many types, the triple $\mathrm{cm}_3(T) = (P(T), L(T), \mathrm{NPL}(T))$ has one of the following values:
(1) $(2^\omega, 2^\omega, \lambda)$, *where* $\lambda \in \omega \cup \{\omega, 2^\omega\}$;
(2) $(0, 0, 2^\omega)$;
(3) $(\lambda_1, \lambda_2, 2^\omega)$, *where* $\lambda_1 \geq 1$, $\lambda_1, \lambda_2 \in \omega \cup \{\omega, 2^\omega\}$.
All these values have realizations in the class $\mathcal{T}_c$.

[1] R.A. Popkov, S.V. Sudoplatov, Distributions of countable models of complete theories with continuum many types, Siberian Electronic Mathematical Reports. 2015. Vol. 12. P. 267–291.

Classification of countable models with respect to two basic characteristics

We consider two basic characteristics for the classification of countable models of a theory:

- Rudin–Keisler preorders $\leq_{\mathrm{RK}}$ for isomorphism types of almost prime models:
$$\mathcal{M}(A) \leq_{\mathrm{RK}} \mathcal{M}(B) \Leftrightarrow \mathcal{M}(B) \text{ realizes } \mathrm{tp}(A);$$
- distributions of limit models over equivalence classes of almost prime models.

We obtain a classification for the class of small theories, with respect to these characteristics.

The classification is generalized for the class $\mathcal{T}_c$ (with R.A. Popkov).

Notations

- $\sim_{RK} \rightleftharpoons \leq_{\mathrm{RK}} \cap \leq_{\mathrm{RK}}^{-1}$;
- $\widetilde{\mathbf{M}}$ is the $\sim_{RK}$-class containing the isomorphism type $\mathbf{M}$ for a prime model over a finite set;
- $\mathrm{IL}(\widetilde{\mathbf{M}})$ is the number of limit models being unions of elementary chains of models with isomorphism types in $\widetilde{\mathbf{M}}$.

Characterization of $I(T,\omega) < \omega$ with respect to limit models[2] [3]

Theorem

For any countable complete theory T, the following conditions are equivalent:
(1) $I(T,\omega) < \omega$;
(2) T is small, $|\mathrm{RK}(T)| < \omega$ and $\mathrm{IL}(\widetilde{\mathbf{M}}) < \omega$ for any $\widetilde{\mathbf{M}} \in \mathrm{RK}(T)/\sim_{\mathrm{RK}}$.

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[2] Sudoplatov S. V. Complete theories with finitely many countable models. I // Algebra and Logic. — 2004. — Vol. 43, No. 1. — P. 62–69.

[3] Sudoplatov S. V. Classification of Countable Models of Complete Theories. — Novosibirsk : NSTU, 2014, 2018.

Characterization of $I(T,\omega) < \omega$ with respect to limit models

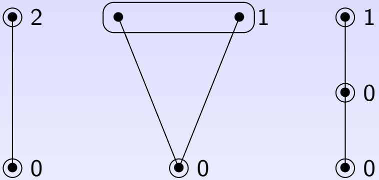
If (1) or (2) holds then T possesses the following properties:
(a) $\mathrm{RK}(T)$ has a least element $\mathbf{M}_0$ (an isomorphism type of a prime model) and $\mathrm{IL}(\widetilde{\mathbf{M}_0}) = 0$;
(b) $\mathrm{RK}(T)$ has a greatest $\sim_{\mathrm{RK}}$-class $\widetilde{\mathbf{M}_1}$ (a class of isomorphism types of all prime models over realizations of powerful types) and $|\mathrm{RK}(T)| > 1$ implies $\mathrm{IL}(\widetilde{\mathbf{M}_1}) \geq 1$;
(c) if $|\widetilde{\mathbf{M}}| > 1$ then $\mathrm{IL}(\widetilde{\mathbf{M}}) \geq 1$.
Moreover, the following *decomposition formula* holds:

$$I(T,\omega) = |\mathrm{RK}(T)| + \sum_{i=0}^{|\mathrm{RK}(T)/\sim_{\mathrm{RK}}|-1} \mathrm{IL}(\widetilde{\mathbf{M}_i}), \qquad (1)$$

where $\widetilde{\mathbf{M}_0}, \ldots, \widetilde{\mathbf{M}_{|\mathrm{RK}(T)/\sim_{\mathrm{RK}}|-1}}$ are all elements of the partially ordered set $\mathrm{RK}(T)/\sim_{\mathrm{RK}}$.

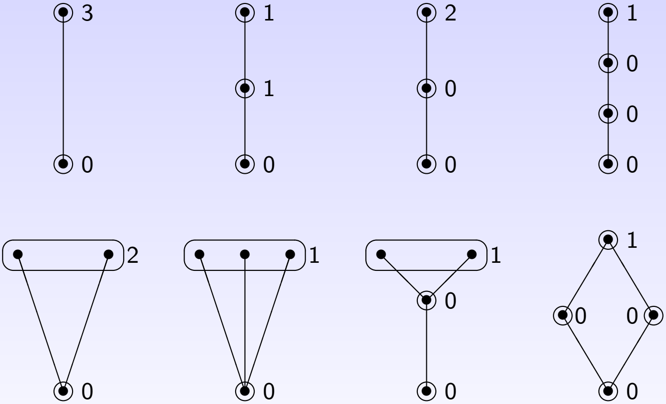
Examples of diagrams for Ehrenfeucht theories

$I(T, \omega) = 3$



$I(T, \omega) = 4$

Examples of diagrams for Ehrenfeucht theories



$$I(T, \omega) = 5$$

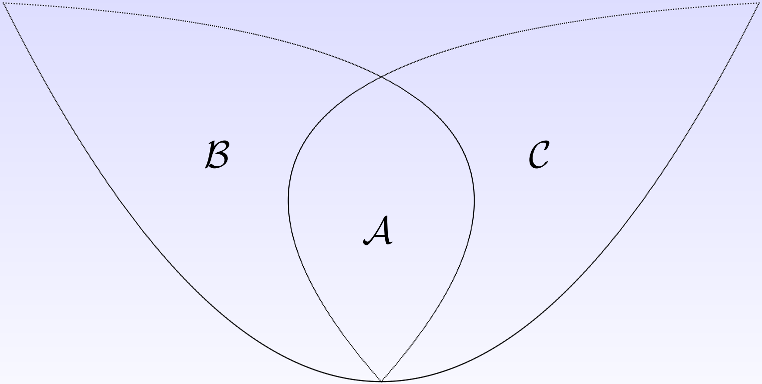
Realizing basic characteristics for the classification, we use syntactic generalizations of semantic Jonsson–Fraïssé–Hrushovski–Herwig generic constructions, based on syntactic amalgams.

Let $\Phi(A)$, $\Psi(B)$, $\mathrm{X}(C)$ be types (diagrams) in a class $\mathbf{T}_0$, describing links between elements in finite sets A, B, C respectively, with some (maybe empty) extra-information, and such that $\Phi(A) \subseteq \Psi(B) \cap \mathrm{X}(C)$.

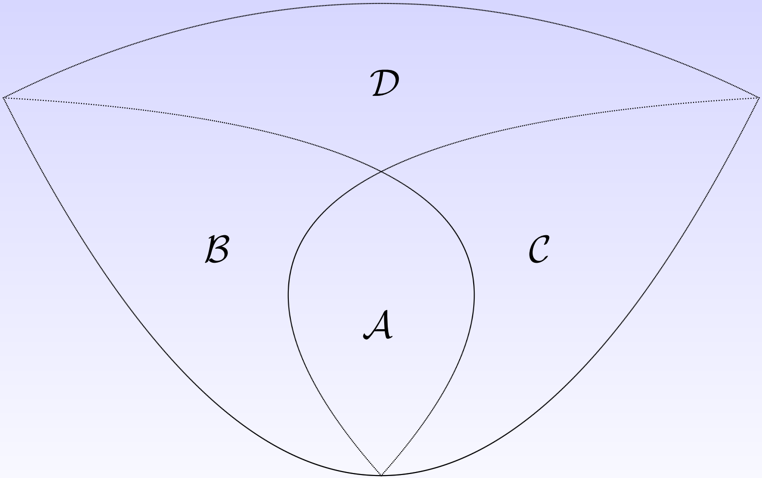
A (*syntactic*) *amalgam* of $\Psi(B)$ and $\mathrm{X}(C)$ over $\Phi(A)$ is a diagram $\Theta(D) \in \mathbf{T}_0$ such that $\Theta(D) \supseteq \Psi(B) \cup \mathrm{X}(C)$.

In particular, these diagrams can contain only inner descriptions for finite structures $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ with universes A, B, C. In such a case, a structure $\mathcal{D}$ with a universe D is a (*semantic*) amalgam of $\mathcal{B}$ and $\mathcal{C}$ over $\mathcal{A}$.

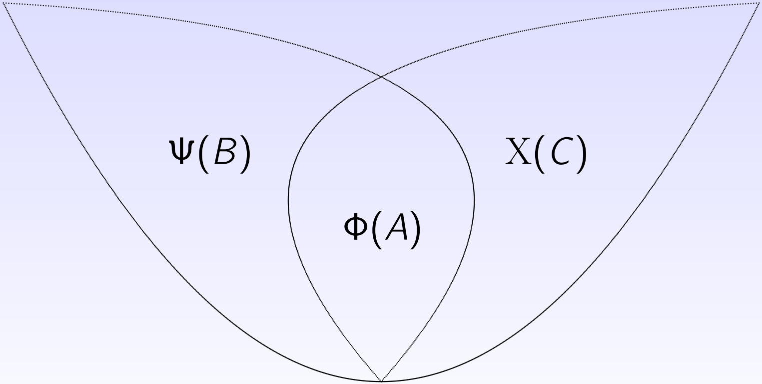
Semantic amalgam



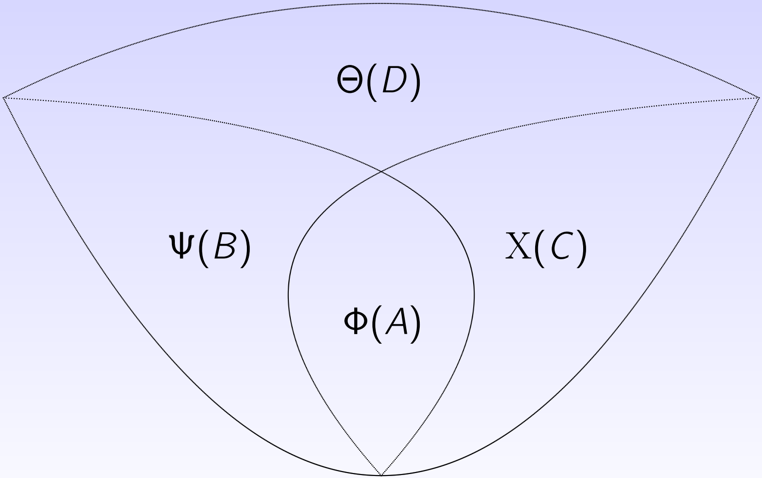
Semantic amalgam



Syntactic amalgam



Syntactic amalgam



Generic structures

We construct a *generic* structure $\mathcal{M}$ step-by-step using a given class $\mathbf{T}_0$ of diagrams and of their amalgams such that all diagrams in $\mathbf{T}_0$ are represented in $\mathcal{M}$:

(1) every finite set A_0 in the universe M of $\mathcal{M}$ is extensible to a finite set $A \subseteq M$ with a diagram $\Phi(A) \in \mathbf{T}_0$ satisfied in $\mathcal{M}$: $\mathcal{M} \models \Phi(A)$;

(2) if $A \subseteq M$ is a finite set, $\Phi(A), \Psi(B) \in \mathbf{T}_0$, $\mathcal{M} \models \Phi(A)$ and $\Phi(A) \leqslant \Psi(B)$ (where $\leqslant$ is a given upward directed partial order for $\mathbf{T}_0$ coordinated with $\subseteq$), then there exists a set $B' \subseteq M$ such that $A \subseteq B'$ and $\mathcal{M} \models \Psi(B')$.

Generic structures

Every finite part of the extra-information should be realized on some step: if $\exists x \varphi(x) \in \Phi(A)$ then there are $B \supseteq A$ with $\Psi(B) \supseteq \Phi(A)$ and an element $b \in B$ such that $\varphi(b) \in \Psi(B)$.

Finite steps approximate the required generic structure.

Generic structures and generic theories

We form a required theory (with desirable properties) introducing an appropriate class $\mathbf{T}_0$ of (in)complete diagrams.

If the process is organized uniformly then diagrams $\Phi(A) \in \mathbf{T}_0$ force complete types $\mathrm{tp}(A)$.

It is natural to take diagrams with a minimal information including atomic links by predicates and operations.

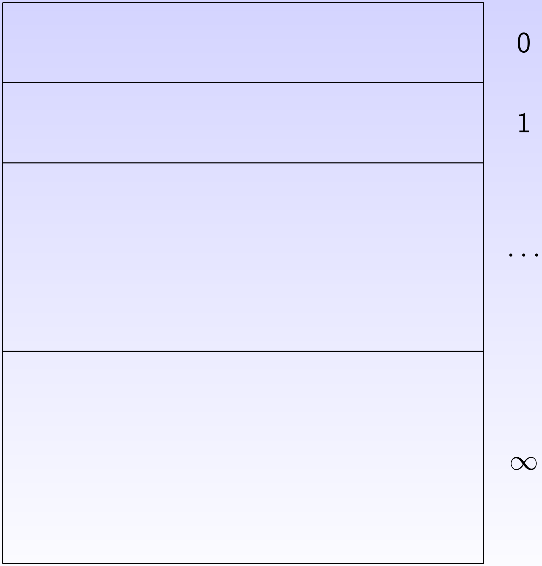
We put an information via the class $\mathbf{T}_0$ and obtain a *generic* theory T such that models of T realize the information I. Here, step-by-step we construct simultaneously a syntactic object T collecting the required information I and a semantic object $\mathcal{M}$ realizing I.

Generic constructions forcing given distributions of countable models

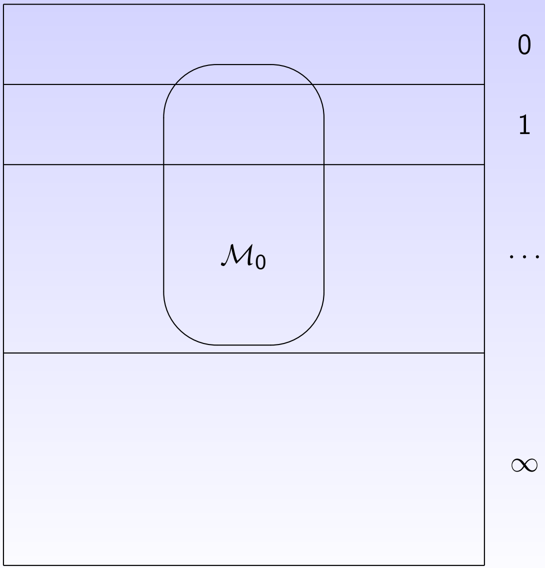
There are natural examples but they do not cover all characteristics.

We illustrate the mechanism for realizations of basic characteristics on the following examples.

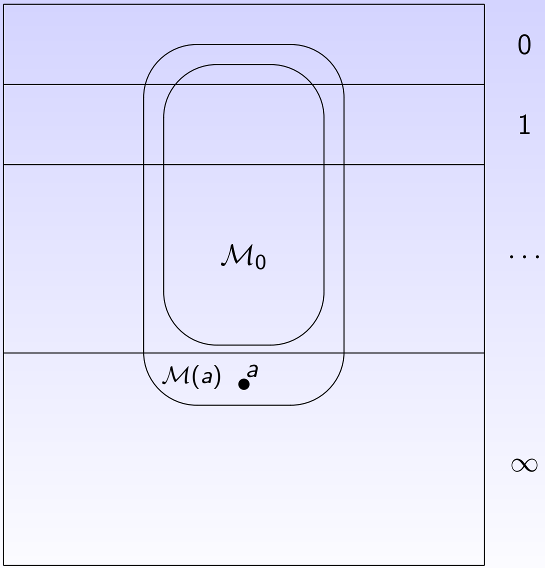
$I(T, \omega) = 3$, $P(T) = 2$, $L(T) = 1$



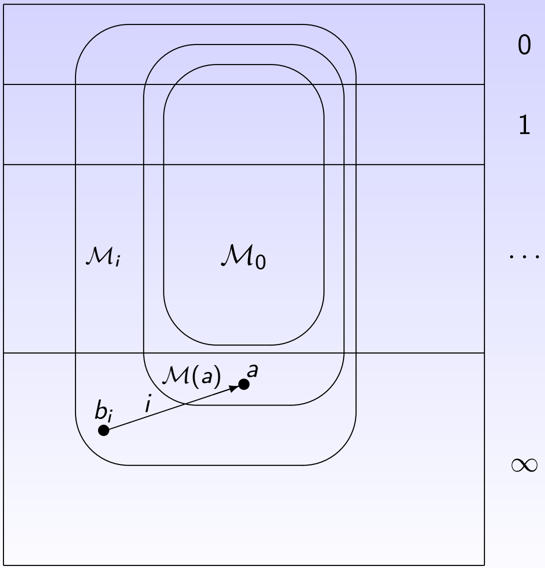
$I(T,\omega)=3$, $P(T)=2$, $L(T)=1$



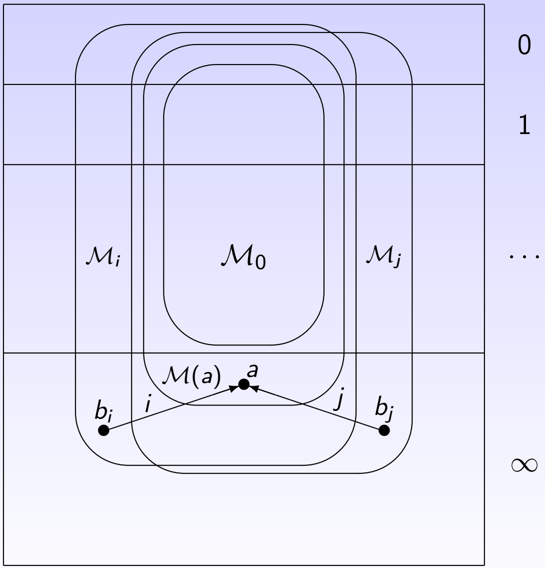
$I(T,\omega)=3$, $P(T)=2$, $L(T)=1$



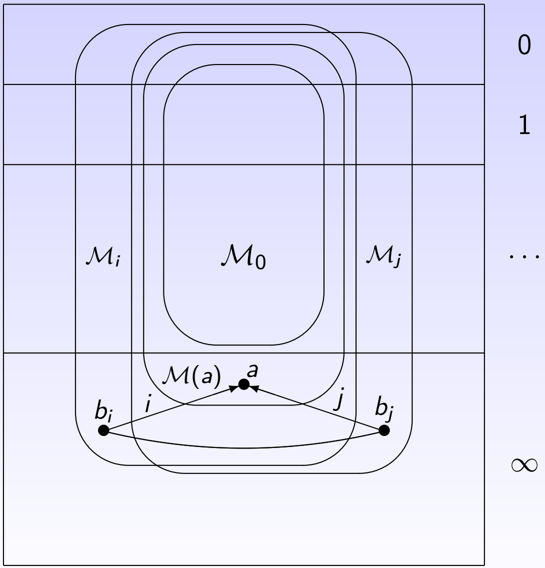
$I(T,\omega) = 3$, $P(T) = 2$, $L(T) = 1$



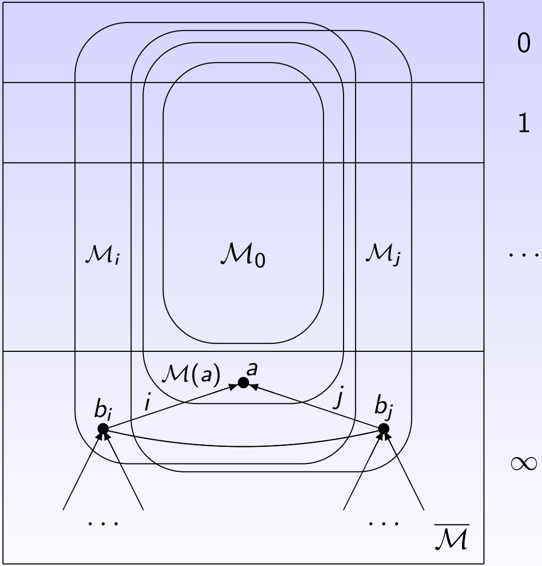
$I(T,\omega) = 3,\ P(T) = 2,\ L(T) = 1$



$I(T,\omega) = 3,\ P(T) = 2,\ L(T) = 1$



$I(T,\omega)=3,\ P(T)=2,\ L(T)=1$



$I(T,\omega)=3$, $P(T)=2$, $L(T)=1$

We consider countably many disjoint colors $0,1,\dots,n,\dots$ for elements and directed links by colored arcs also with countably many disjoint colors ordered by colors of elements.

For the theory T we have a prime model $\mathcal{M}_0$ whose all elements have finite colors.

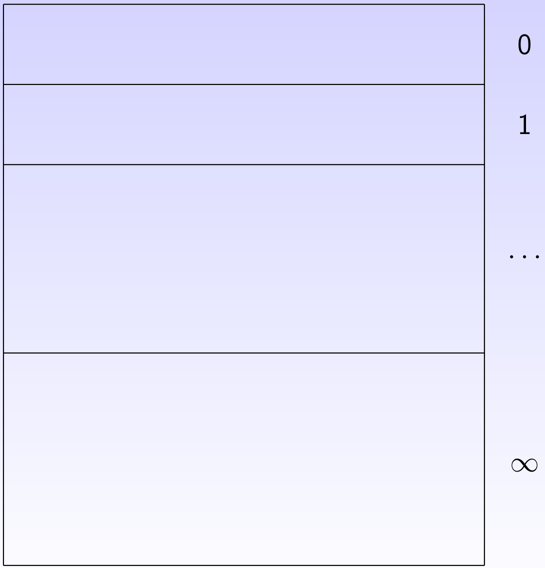
Furthermore, there is a prime model $\mathcal{M}(a)$ over a realization a of the type describing the infinite color.

The model $\mathcal{M}(a)$ has countably many essentially distinct extensions $\mathcal{M}(b_i)$ by arcs of colors i.

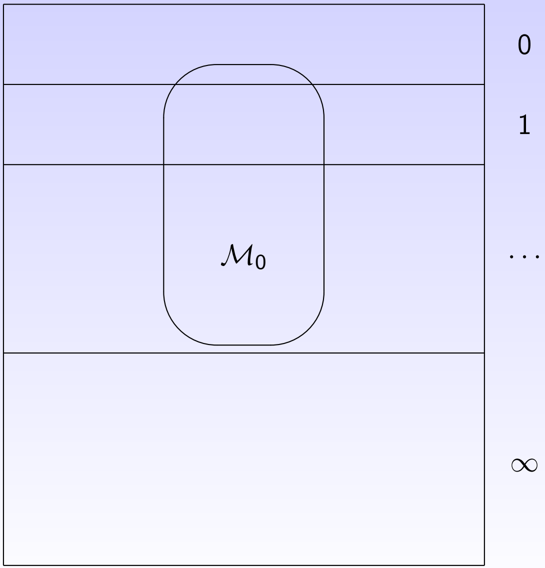
We introduce "bridges", i.e., principal edges $[b_i,b_j]$, guaranteing that any i-extension $\mathcal{M}_i=\mathcal{M}(b_i)$ is equal to a j-extension $\mathcal{M}_j=\mathcal{M}(b_j)$.

Having these bridges we obtain unique limit model $\overline{\mathcal{M}}$ (together with two non-isomorphic almost prime models $\mathcal{M}_0$ and $\mathcal{M}(a)$).

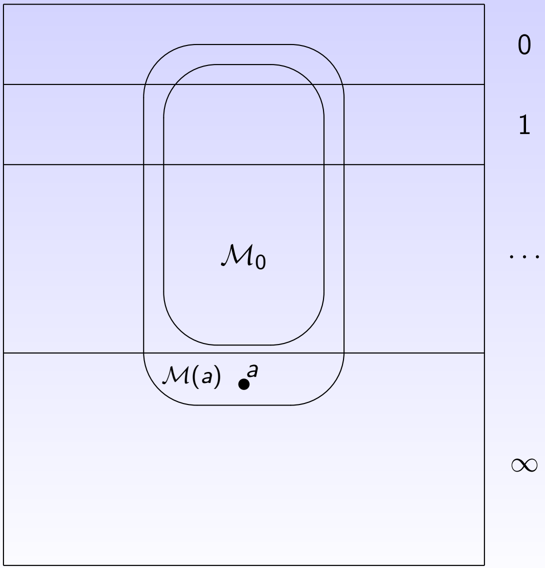
$I(T,\omega)=4$, $P(T)=2$, $L(T)=2$



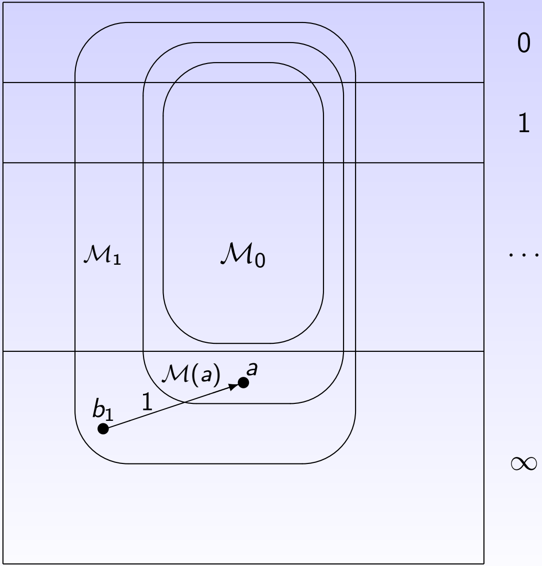
$I(T, \omega) = 4$, $P(T) = 2$, $L(T) = 2$



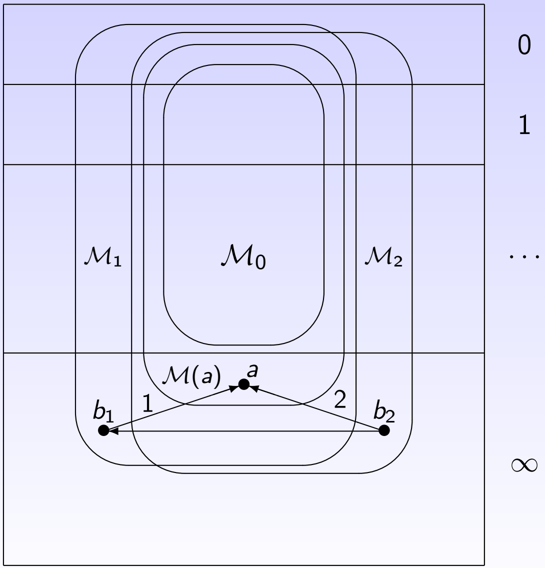
$I(T,\omega)=4$, $P(T)=2$, $L(T)=2$



$I(T,\omega)=4$, $P(T)=2$, $L(T)=2$



$I(T,\omega)=4,\ P(T)=2,\ L(T)=2$



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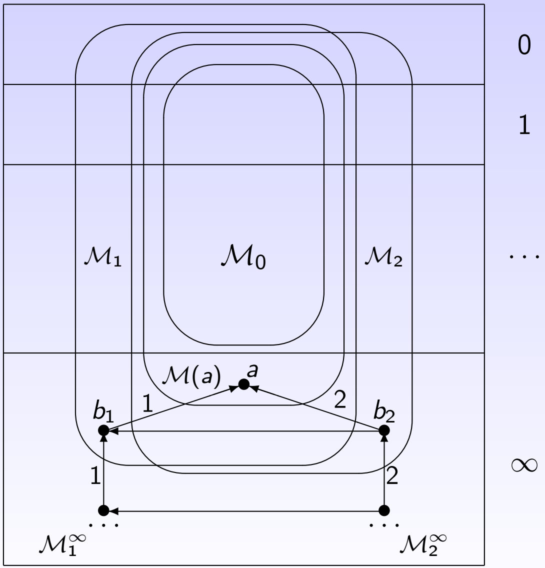


$I(T,\omega) = 4$, $P(T) = 2$, $L(T) = 2$

Basing on the example of theory with three countable models we consider again countably many disjoint colors $0, 1, \ldots, n, \ldots$ for elements and directed links by colored arcs also with countably many disjoint colors ordered by colors of elements.

For the theory T we have a prime model $\mathcal{M}_0$ whose all elements have finite colors.

Furthermore, there is a prime model $\mathcal{M}(a)$ over a realization a of the type describing the infinite color.

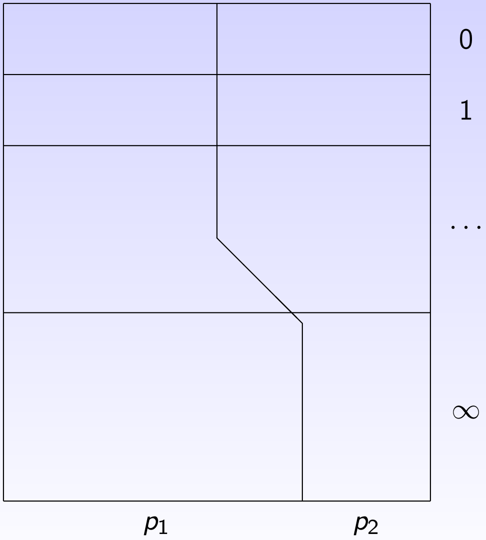
The model $\mathcal{M}(a)$ has two essentially distinct extensions $\mathcal{M}_1 = \mathcal{M}(b_1)$ and $\mathcal{M}_2 = \mathcal{M}(b_2)$ by arcs of colors 1 and 2 respectively.

$I(T,\omega)=4$, $P(T)=2$, $L(T)=2$

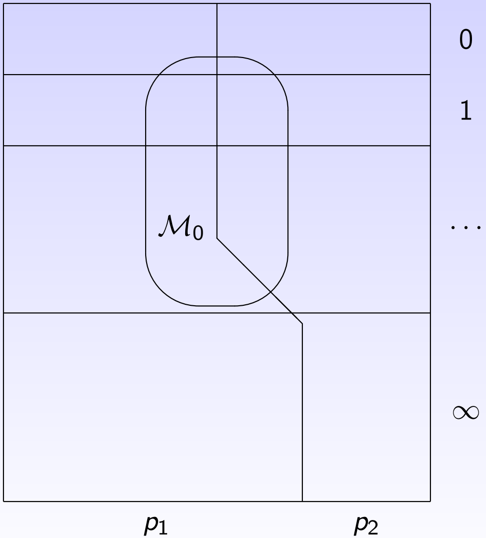
We introduce arcs (b_2, b_1) guaranteing that any 2-extension $\mathcal{M}_2$ includes a 1-extension $\mathcal{M}_1$ but not vice versa.

Having these arcs we obtain two limit models $\mathcal{M}_1^\infty$ and $\mathcal{M}_2^\infty$ corresponding to elementary chains of 1-extensions and of 2-extensions (together with two non-isomorphic almost prime models $\mathcal{M}_0$ and $\mathcal{M}(a)$). Here $\mathcal{M}_2^\infty$ is saturated.

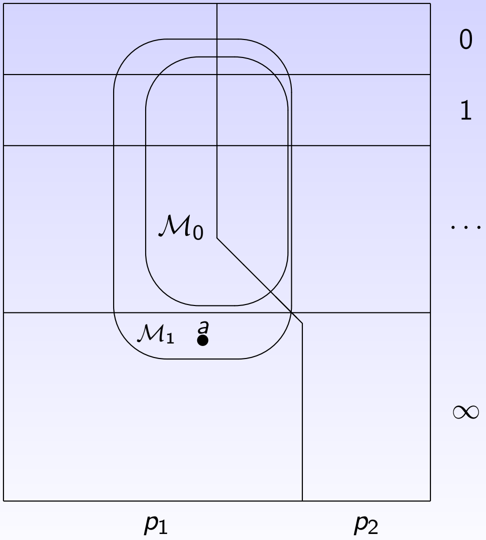
$I(T,\omega) = 4$, $P(T) = 3$, $L(T) = 1$, linear $\leq_{\mathrm{RK}}$



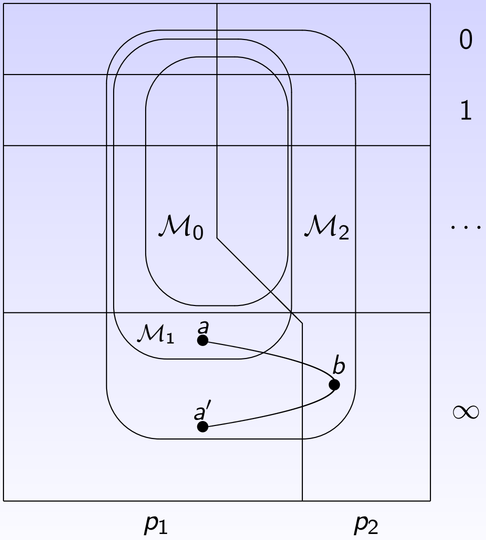
$I(T,\omega)=4$, $P(T)=3$, $L(T)=1$, linear $\leq_{\mathrm{RK}}$



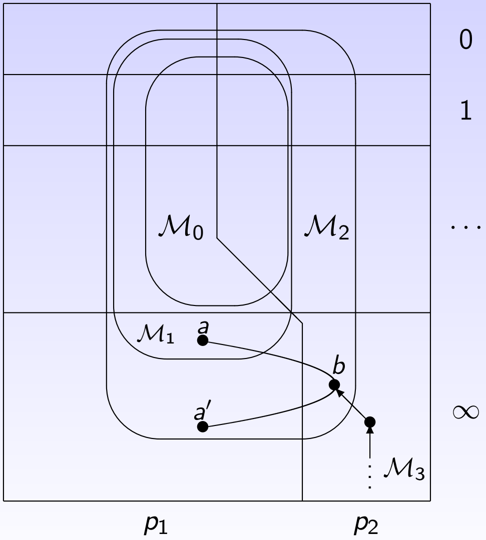
$I(T,\omega)=4$, $P(T)=3$, $L(T)=1$, linear $\leq_{\rm RK}$



$I(T,\omega)=4$, $P(T)=3$, $L(T)=1$, linear $\leq_{\mathrm{RK}}$



$I(T,\omega) = 4$, $P(T) = 3$, $L(T) = 1$, linear $\leq_{\mathrm{RK}}$



$I(T,\omega)=4$, $P(T)=3$, $L(T)=1$, linear $\leq_{\rm RK}$

Basing on the previous examples, for two disjoint infinite unary predicates P_0 and P_1, we consider countably many disjoint colors $0,1,\ldots,n,\ldots$ for elements and directed links by colored arcs also with countably many disjoint colors ordered by colors of elements. Thus we have two non-isolated 1-types p_1 and p_2 realizing predicates P_0 and P_1 by elements of infinite color.

For the theory T we have a prime model $\mathcal{M}_0$ whose all elements have finite colors.

Furthermore, there is a prime model $\mathcal{M}_1=\mathcal{M}(a)$ over a realization a of the type p_1 such that $\mathcal{M}_1$ has a unique realization of p_1.

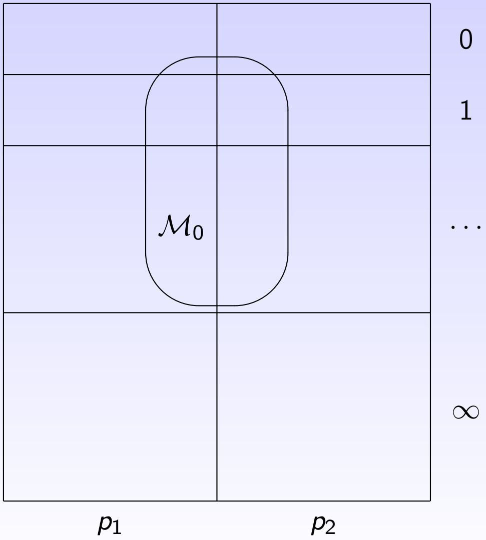
The model $\mathcal{M}(a)$ has an extension $\mathcal{M}_2=\mathcal{M}(b)$, where b is a realization of p_2. Moreover, having a and a realization $a'\neq a$ of p_1 we obtain a model isomorphic to $\mathcal{M}_2$, w.l.o.g. $\mathcal{M}(b)=\mathcal{M}(a,a')$.

Extending the model $\mathcal{M}_2$ by principal arcs we get unique limit model $\mathcal{M}_3$.

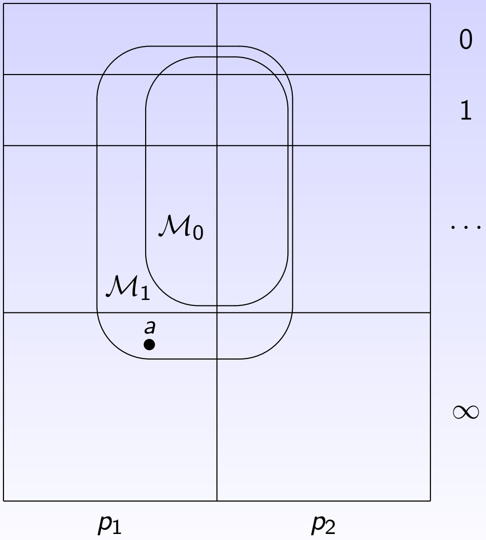
$I(T,\omega)=4$, $P(T)=3$, $L(T)=1$, non-linear $\leq_{\rm RK}$

		0
		1
		$\ldots$
		∞
p_1	p_2	

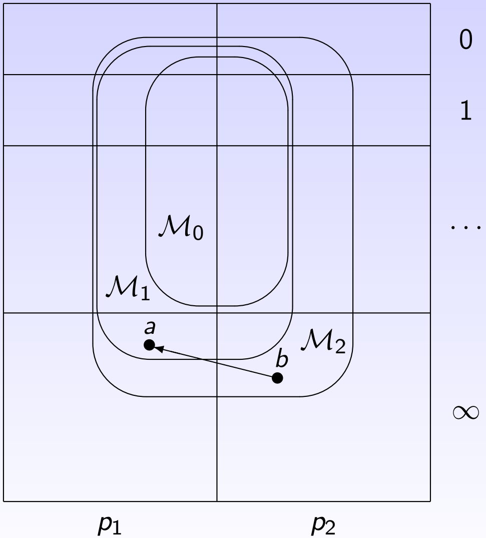
$I(T,\omega)=4$, $P(T)=3$, $L(T)=1$, non-linear $\leq_{\mathrm{RK}}$



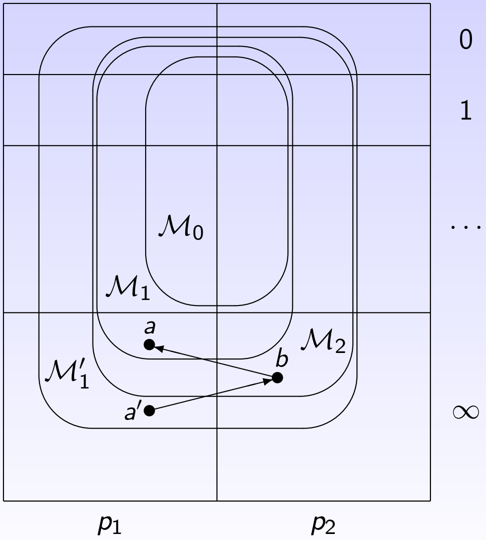
$I(T,\omega) = 4$, $P(T) = 3$, $L(T) = 1$, non-linear $\leq_{\mathrm{RK}}$



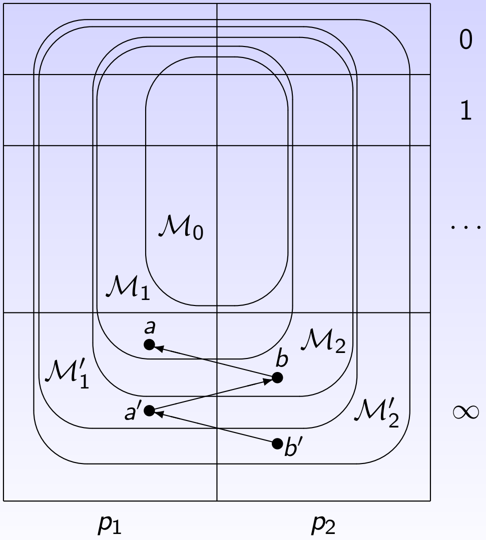
$I(T,\omega)=4$, $P(T)=3$, $L(T)=1$, non-linear $\leq_{\rm RK}$



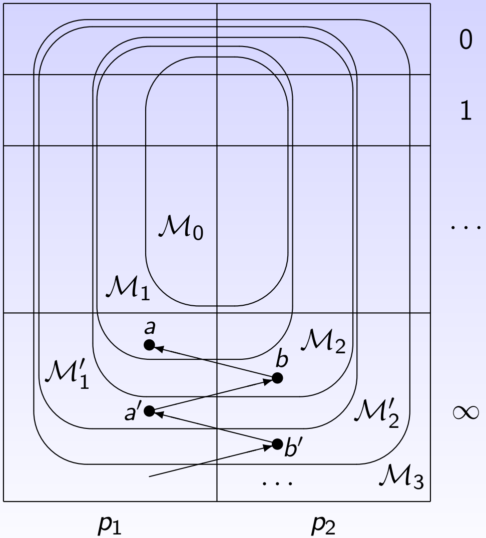
$I(T,\omega) = 4$, $P(T) = 3$, $L(T) = 1$, non-linear $\leq_{\mathrm{RK}}$



$I(T,\omega)=4$, $P(T)=3$, $L(T)=1$, non-linear $\leq_{\mathrm{RK}}$



$I(T,\omega) = 4$, $P(T) = 3$, $L(T) = 1$, non-linear $\leq_{\mathrm{RK}}$



$I(T,\omega) = 4$, $P(T) = 3$, $L(T) = 1$, non-linear $\leq_{\rm RK}$

Again for two disjoint infinite unary predicates P_0 and P_1, we consider countably many disjoint colors $0, 1, \ldots, n, \ldots$ for elements and directed links by colored arcs also with countably many disjoint colors ordered by colors of elements. We have two non-isolated 1-types p_1 and p_2 realizing predicates P_0 and P_1 by elements of infinite color.

For the theory T we have a prime model $\mathcal{M}_0$ whose all elements have finite colors.

Furthermore, there is a prime model $\mathcal{M}_1 = \mathcal{M}(a)$ over a realization a of the type p_1.

The model $\mathcal{M}(a)$ has a proper extension $\mathcal{M}_2 = \mathcal{M}(b)$ by a principal arc (b, a), where b is a realization of p_2 and $\mathcal{M}_2$ in not isomorphic to $\mathcal{M}_1$.

$I(T,\omega)=4$, $P(T)=3$, $L(T)=1$, non-linear $\leq_{\rm RK}$

Then the model $\mathcal{M}_2$ has a proper extension $\mathcal{M}'_1=\mathcal{M}(a')$ by a principal arc (a',b), where a' is a realization of p_1, $\mathcal{M}'_1$ has a proper extension $\mathcal{M}'_2=\mathcal{M}(b')$ by a principal arc (b',a'), where b' is a realization of p_2, etc.

Extending the chain of almost prime models we get unique limit model $\mathcal{M}_3$.

We consider a series of derivative objects used for a classification of structures and their elementary theories:

- Rudin–Keisler preorders and distribution functions for limit models of a given theory, producing spectra of countable models (with B.S. Baizhanov, B.Sh. Kulpeshov, and V.V. Verbovskiy);
- Topologies, closures, generating sets, e-spectra, ranks, and approximations for families of theories with respect to P-operators and E-operators (with B.Sh. Kulpeshov, N.D. Markhabatov, and In.I. Pavlyuk).
- Formulas for families of theories, and of their characteristics (with In.I. Pavlyuk).
- Arities of theories, their dynamics and characteristics with respect to closures.

Pseudo-supersimple theories [4] [5] [6]

Definition (A. Tsuboi)

A stable theory T is *pseudo-superstable* if T fails to have the infinite weight.

Similarly we say that a simple theory T is *pseudo-supersimple* if T fails to have the infinite weight.
We generalize both the Tsuboi theorem for pseudo-superstable theories and the Kim theorem for simple theories, obtaining

Theorem (B.S. Baizhanov, S.V. Sudoplatov, V.V. Verbovskiy, 2012)

Let T be a union of pseudo-supersimple theories T_n, where $T_n \subseteq T_{n+1}$, $n \in \omega$. Then $I(T, \omega) = 1$ or $I(T, \omega) \geq \omega$.

[4] A. Tsuboi, Countable models and unions of theories, J. Math. Soc. Japan. 1986. Vol. 38, No. 3. P. 501–508.

[5] B. Kim , On the number of countable models of a countable supersimple theory, J. London Math. Soc. 1999. Vol. 60, No. 2. P. 641–645.

[6] B. S. Baizhanov, S. V. Sudoplatov, V. V. Verbovskiy, Conditions for non-symmetric relations of semi-isolation, Siberian Electronic Mathematical Reports. 2012. Vol. 9. P. 161–184.

Weakly o-minimal structures[7]

Definition

A *weakly o-minimal structure* is a linearly ordered structure $\mathcal{M} = \langle M, =, <, \ldots \rangle$ such that any definable (with parameters) subset of the structure M is a finite union of convex sets in $\mathcal{M}$.

We recall that such a structure $\mathcal{M}$ is said to be *o-minimal* if any definable (with parameters) subset of $\mathcal{M}$ is the union of finitely many intervals and points in $\mathcal{M}$.

[7] H.D. Macpherson, D. Marker, and C. Steinhorn, Weakly o-minimal structures and real closed fields // Transactions of The American Mathematical Society, 352 (2000), pp. 5435–5483.

Weakly orthogonal types[8]

Definition

Assuming that $\mathcal{M}$ is an $|A|^+$-saturated weakly o-minimal structure, $A, B \subseteq M$, and $p, q \in S^1(A)$ are non-algebraic types, we say that p is not *weakly orthogonal* to q ($p \not\perp^w q$) if there are an A-definable formula $H(x, y)$, $a \in p(\mathcal{M})$, and $b_1, b_2 \in q(\mathcal{M})$ such that $b_1 \in H(\mathcal{M}, a)$ and $b_2 \notin H(\mathcal{M}, a)$.

Lemma (B.S. Baizhanov)

The relation $\not\perp^w$ of the weak non-orthogonality is an equivalence relation on $S^1(A)$.

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[8] B.S. Baizhanov, Expansion of a model of a weakly o-minimal theory by a family of unary predicates // The Journal of Symbolic Logic, 66 (2001), pp. 1382–1414.

Quite o-minimal theories[9]

Definition

We say that p is not *quite orthogonal* to q ($p \not\perp^q q$) if there is an A-definable bijection $f : p(M) \to q(M)$.
We say that a weakly o-minimal theory is *quite o-minimal* if the relations of weak and quite orthogonality coincide for 1-types over arbitrary sets of models of the given theory.

[9] B.Sh. Kulpeshov, Convexity rank and orthogonality in weakly o-minimal theories // News of the National Academy of Sciences of the Republic of Kazakhstan, physical and mathematical series, 227 (2003), pp. 26–31.

Quite *o*-minimal theories with few countable models[10]

We say that a quite o-minimal theory T has *few countable models* if T has fewer than 2^ω countable models up to isomorphisms.

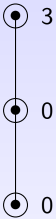
Theorem

Let T be a quite o-minimal theory in a countable language. Then either T has 2^ω countable models or T has exactly $3^k \cdot 6^s$ countable models, where k and s are natural numbers. Moreover, for any $k, s \in \omega$ there is a quite o-minimal theory T with exactly $3^k \cdot 6^s$ countable models.

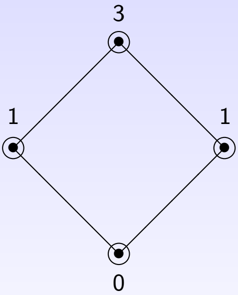
Theorem above generalizes the known Mayer's theorem on countable models of *o*-minimal theories: L.L. Mayer, Vaught's conjecture for o-minimal theories // The Journal of Symbolic Logic, 53 (1988), pp. 146–159.

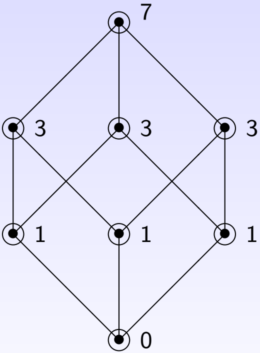
[10] B.Sh. Kulpeshov, S.V. Sudoplatov, Vaught's conjecture for quite o-minimal theories // Annals of Pure and Applied Logic, 2017. Vol. 168, N 1. P. 129-149.

Quite o-minimal theories with $I(T,\omega) = 3$ and $I(T,\omega) = 6$

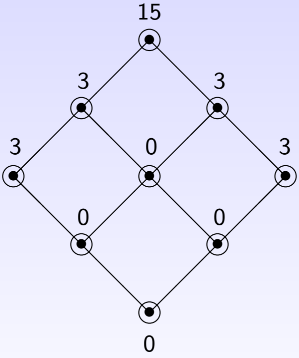


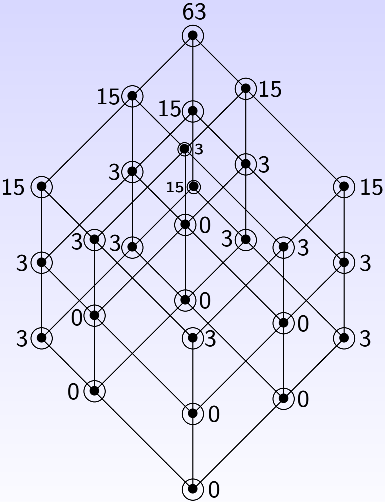
Quite o-minimal theories with $I(T,\omega)=3^2$ and $I(T,\omega)=3^3$



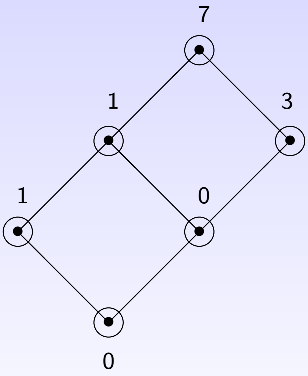


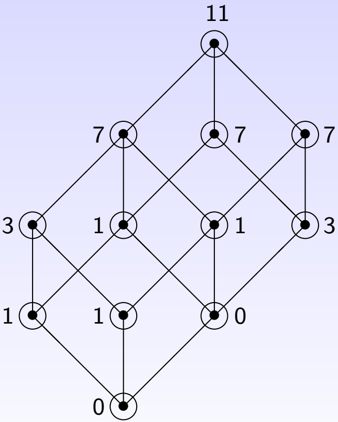
Quite o-minimal theories with $I(T,\omega)=6^2$ and $I(T,\omega)=6^3$



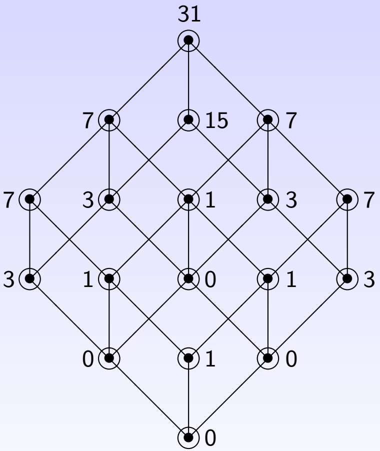


Quite o-minimal theories with $I(T,\omega)=3\cdot 6$ and $I(T,\omega)=3^2\cdot 6$





Quite o-minimal theories with $I(T,\omega) = 3 \cdot 6^2$



Decomposition formula (with B.Sh. Kulpeshov) [11] [12]

Applying Theorem on numbers of countable models for quite *o*-minimal theories we have the following representation of Decomposition formula (1):

$$3^k \cdot 6^s = 2^k \cdot 3^s + \sum_{t=0}^{k} \sum_{m=0}^{s} 2^{s-m} \cdot (2^t \cdot 4^m - 1) \cdot C_k^t \cdot C_s^m.$$

[11] B.Sh. Kulpeshov, S.V. Sudoplatov, Distributions of countable models of quite o-minimal Ehrenfeucht theories, Eurasian Mathematical Journal. 2020. Vol. 11, No. 3. P. 66–78.

[12] S.V. Sudoplatov, Distributions of countable models of disjoint unions of Ehrenfeucht theories, Lobachevskii Journal of Mathematics. 2021. Vol. 42, No. 1. P. 195–205.

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P-combinations

Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \rightleftharpoons \bigcup\limits_{i\in I} \mathcal{A}_i$ expanded by the predicates P_i is the *P-union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the *P-operator*. The structure $\mathcal{A}_P$ is called the *P-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P|_{A_i})|_{\Sigma(\mathcal{A}_i)}$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

P-combinations

Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}'|_{P_i})|_{\Sigma(\mathcal{A}_i)}$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write $\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}'|_{\bigcap\limits_{i\in I} \overline{P_i}}$, maybe applying Morleyzation. Moreover, we write $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$ for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

P-combinations

Note that if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$, $\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_P = \mathrm{Th}(\mathcal{A}_P)$, being *$P$-combination* of T_i, which is denoted by $\mathrm{Comb}_P(T_i)_{i \in I}$. In general, for non-disjoint case, the theory T_P will be also called a *P-combination* of the theories T_i, but in such a case we will keep in mind that this P-combination is constructed with respect (and depending) to the structure $\mathcal{A}_P$, or, equivalently, with respect to any/some $\mathcal{A}' \equiv \mathcal{A}_P$.

E-combinations

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ being the *E-union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the *E-operator*. The structure $\mathcal{A}_E$ is also called the *E-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E|_{A_i})|_{\Sigma(\mathcal{A}_i)}$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes. The E-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_E = \mathrm{Th}(\mathcal{A}_E)$, being *$E$-combination* of T_i, which is denoted by $\mathrm{Comb}_E(T_i)_{i\in I}$ or by $\mathrm{Comb}_E(\mathcal{T})$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

E-combinations

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the *E-representability*.

E-combinations

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by $e\text{-}\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-}\mathrm{Sp}(\mathcal{A}') \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by $e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$.

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

E-closures

Definition. Let $\overline{\mathcal{T}}$ be the class of all complete elementary theories of relational languages. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i \in I}$, $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

The operator Cl_E of E-closure can be naturally extended to the classes $\mathcal{T} \subset \overline{\mathcal{T}}$ as follows: $\mathrm{Cl}_E(\mathcal{T})$ is the union of all $\mathrm{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0 \subseteq \mathcal{T}$.

For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$.

Proposition

If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $T \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an *accumulation point* for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any formula $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.

Least generating sets

Theorem

If $\mathcal{T}_0'$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:
(1) $\mathcal{T}_0'$ is the least generating set for $\mathcal{T}_0$;
(2) $\mathcal{T}_0'$ is a minimal generating set for $\mathcal{T}_0$;
(3) any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0')_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0')_\varphi = \{T\}$;
(4) any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.

Ranks for families of theories

For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}(\mathcal{T}) = -1$, for finite nonempty families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) = 0$, and for infinite families $\mathcal{T}$ — $\mathrm{RS}(\mathcal{T}) \geq 1$.
For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.
If α is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.
We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.
If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

Totally transcendental families of theories

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.
If $\mathcal{T}$ is totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Theorem

For any family $\mathcal{T}$ with $|\Sigma(\mathcal{T})| \leq \omega$ the following conditions are equivalent:
(1) $|\mathrm{Cl}_E(\mathcal{T})| = 2^\omega$;
(2) e-$\mathrm{Sp}(\mathcal{T}) = 2^\omega$;
(3) $\mathrm{RS}(\mathcal{T}) = \infty$;
(4) there exists a 2-tree of sentences φ for s-definable properties P_φ.

Families of all theories of given languages

Theorem (N.D. Markhabatov, S.V. Sudoplatov)

For any language Σ either $\mathrm{RS}(\mathcal{T}_\Sigma)$ is finite, if Σ consists of finitely many 0-ary and unary predicates, and finitely many constant symbols, or $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$, otherwise.

Approximations of theories of abelian groups

Theorem (In.I. Pavlyuk< S.V. Sudoplatov)

For any theory T of abelian groups the following conditions are equivalent:
(1) T is pseudofinite;
(2) T has some infinite $\alpha_{p,n}$, or some $\beta_p = \gamma_p = \omega$, or $\varepsilon = 1$, moreover, for all nonzero values β_p and γ_p, $\beta_p = \gamma_p = \omega$;
(3) T has infinite models, and all nonzero values β_p and γ_p imply $\beta_p = \gamma_p = \omega$.

Corollary

If a theory T of an abelian group has a positive natural value β_p or γ_p then models of T are not pseudofinite.

Since $\mathrm{Th}(\mathbb{Z})$ has values $\gamma_p = 1$, the group $\mathbb{Z}$ is not pseudofinite.

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- S.V. Sudoplatov, Formulas and properties, their links and characteristics, Mathematics. 2021, Vol. 9, Issue 12. 1391. 16 pp.
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Definition. Let Σ be a language, $\varphi \rightleftharpoons \varphi(\overline{x})$ be a formula in $F(\Sigma)$, P_s be a subclass of the class $K(\Sigma)$ of all structures $\mathcal{A}$ in the language Σ. We say that $\varphi(\overline{x})$ *partially* (respectively, *totally*) *satisfies* P_s, denoted by $\varphi \rhd_{\rm ps} P_s$ or $\varphi \rhd_s^{\exists} P_s$ ($\varphi \rhd_{\rm ts} P_s$ or $\varphi \rhd_s^{\forall} P_s$), if there are $\mathcal{A} \in P_s$ and $\overline{a} \in A$ (for any $\mathcal{A} \in P_s$ there is $\overline{a} \in A$) such that $\mathcal{A} \models \varphi(\overline{a})$.

If $P_{\rm is}$ is a subclass of the class ${\rm ITK}(\Sigma)$ of isomorphism types for the class $K(\Sigma)$ then we say that $\varphi(\overline{x})$ *partially* (respectively, *totally*) *satisfies* $P_{\rm its}$, denoted by $\varphi\rhd_{\rm pits}P_{\rm its}$ or $\varphi\rhd^{\exists}_{\rm its}P_{\rm its}$ ($\varphi\rhd_{\rm tits}P_{\rm its}$ or $\varphi\rhd^{\forall}_{\rm its}P_{\rm its}$) if $\varphi\rhd_{\rm ps}P_s$ ($\varphi\rhd_{\rm ts}P_s$, where P_s consists of all structures whose isomorphism types belong to $P_{\rm its}$. If P_t is a subset of the set $\mathcal{T}_\Sigma$ of all complete theories in the language Σ then we say that $\varphi(\overline{x})$ *partially* (respectively, *totally*) *satisfies* P_t, denoted by $\varphi\rhd_{\rm pt}P_t$ or $\varphi\rhd^{\exists}_{t}P_t$ ($\varphi\rhd_{\rm tt}P_t$ or $\varphi\rhd^{\forall}_{t}P_t$), if there are $T\in P_t$, $\mathcal{M}\models T$, and $\overline{a}\in M$ (for any $T\in P_t$ there are $\mathcal{M}\models T$ and $\overline{a}\in M$) such that $\mathcal{M}\models\varphi(\overline{a})$.

Semantic and syntactic properties, their links with formulas

For a property P_s we denote by $\mathrm{ITK}(P_s)$ the class of isomorphism types for structures in P_s, and by $\mathrm{Th}(P_s)$ the set $\{T \in \mathcal{T}_\Sigma \mid \mathcal{A} \models T \text{ for some } \mathcal{A} \in P_s\}$. For a property $P_{\rm its}$ we denote by $K(P_{\rm its})$ the class of all structures whose isomorphism types are represented in $P_{\rm its}$, and by $\mathrm{Th}(P_{\rm its})$ the set $\mathrm{Th}(K(P_{\rm its}))$. For a property P_t we denote by $K(P_t)$ the class of all models of theories in P_t, and by $\mathrm{ITK}(P_t)$ the class $\mathrm{ITK}(K(P_t))$.

Semantic and syntactic properties, their links with formulas

Proposition

For any formula $\varphi \in F(\Sigma)$ and properties P_s, $P_{\rm its}$, P_t the following conditions hold:

(1) $\varphi \rhd_{\rm ps} P_s$ iff $\varphi \rhd_{\rm pits} {\rm ITK}(P_s)$, and iff $\varphi \rhd_{\rm pt} {\rm Th}(P_s)$;
(2) $\varphi \rhd_{\rm ts} P_s$ iff $\varphi \rhd_{\rm tits} {\rm ITK}(P_s)$, and iff $\varphi \rhd_{\rm tt} {\rm Th}(P_s)$;
(3) $\varphi \rhd_{\rm pits} P_{\rm its}$ iff $\varphi \rhd_{\rm ps} K(P_{\rm its})$, and iff $\varphi \rhd_{\rm pt} {\rm Th}(P_{\rm its})$;
(4) $\varphi \rhd_{\rm tits} P_{\rm its}$ iff $\varphi \rhd_{\rm ts} K(P_{\rm its})$, and iff $\varphi \rhd_{\rm tt} {\rm Th}(P_{\rm its})$;
(5) $\varphi \rhd_{\rm pt} P_t$ iff $\varphi \rhd_{\rm ps} K(P_t)$, and iff $\varphi \rhd_{\rm pits} {\rm ITK}(P_t)$;
(6) $\varphi \rhd_{\rm tt} P_t$ iff $\varphi \rhd_{\rm ts} K(P_t)$, and iff $\varphi \rhd_{\rm tits} {\rm ITK}(P_t)$.

In the items (3) and (4) the class $K(P_{\rm its})$ can be replaced by a subclass K' such that $\mathrm{ITK}(K')=P_{\rm its}$. Similarly, in the items (5) and (6) the class $K(P_t)$ can be replaced by a subclass K' such that $\mathrm{Th}(K')=P_t$, and independently $\mathrm{ITK}(P_t)$ can be replaced by a subclass K'' such that $\mathrm{Th}(K'')=P_t$.

By Proposition above semantic properties P_s and $P_{\rm its}$ can be naturally transformed into syntactic ones P_t, and vice versa. It means that natural model-theoretic properties such as ω-categoricity, stability, simplicity etc. can be formulated both for theories, for structures and for their isomorphism types. The links between $\rhd$-relations which pointed out in Proposition 6 allow to reduce our consideration to the relations $\rhd_{\rm pt}$ and $\rhd_{\rm tt}$. Besides, for the simplicity we will principally consider sentences φ instead of formulas in general. Reductions of formulas $\psi(\overline{x})$ to sentences use the operators $\psi(\overline{x}) \mapsto \forall\overline{x}\,\psi(\overline{x})$ and $\psi(\overline{x}) \mapsto \exists\overline{x}\,\psi(\overline{x})$.

Formulas, properties and E-closures

Theorem

For any sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P_t \subseteq \mathcal{T}_\Sigma$ the following conditions are equivalent:
(1) $\varphi \rhd_{\mathrm{pt}} P_t$,
(2) $\varphi \rhd_{\mathrm{pt}} \mathrm{Cl}_E(P_t)$,
(3) $\varphi \rhd_{\mathrm{pt}} P'_t$ for any/some P'_t with $\mathrm{Cl}_E(P'_t) = \mathrm{Cl}_E(P_t)$.

Formulas, properties and E-closures

Theorem

For any sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P_t \subseteq \mathcal{T}_\Sigma$ the following conditions are equivalent:
(1) $\varphi \rhd_{\mathrm{tt}} P_t$,
(2) $\varphi \rhd_{\mathrm{tt}} \mathrm{Cl}_E(P_t)$,
(3) $\varphi \rhd_{\mathrm{tt}} P'_t$ for any/some P'_t with $\mathrm{Cl}_E(P'_t) = \mathrm{Cl}_E(P_t)$.

Corollaries

Corollary

For any properties $P_1, P_2 \subseteq \mathcal{T}_\Sigma$ the following conditions hold:
(1) there exists $\varphi \in \mathrm{Sent}(\Sigma)$ such that $\varphi \rhd_{\rm pt} P_1$ and $\neg\varphi \rhd_{\rm pt} P_2$ iff P_1 and P_2 are nonempty and $|P_1 \cup P_2| \geq 2$; in particular, there exists $\varphi \in \mathrm{Sent}(\Sigma)$ such that $\varphi \rhd_{\rm pt} P_1$ and $\neg\varphi \rhd_{\rm pt} P_1$ iff $|P_1| \geq 2$;
(2) there exists $\varphi \in \mathrm{Sent}(\Sigma)$ such that $\varphi \rhd_{\rm tt} P_1$ and $\neg\varphi \rhd_{\rm tt} P_2$ iff $\mathrm{Cl}_E(P_1) \cap \mathrm{Cl}_E(P_2) = \emptyset$.

Corollary

For any nonempty property $P_t \subseteq \mathcal{T}_\Sigma$ the following conditions hold:
(1) the set $\bigcap P_t$ forms a filter $\bigcap P_t/{\equiv}$ on $\{\equiv(\varphi) \mid \varphi \in \mathrm{Sent}(\Sigma)\}$ with respect to $\vdash$;
(2) the filter $\bigcap P_t/{\equiv}$ is principal iff $\bigcap P_t$ is forced by some its sentence, i.e., $\bigcap P_t$ is a finitely axiomatizable theory, which is incomplete for $|P_t| \geq 2$;
(3) the filter $\bigcap P_t/{\equiv}$ is an ultrafilter iff P_t is a singleton.

Ranks and spectra for sentences and properties

Definition. For a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P = P_t \subseteq \mathcal{T}_\Sigma$ we put $\mathrm{RS}_P(\varphi) = \mathrm{RS}(P_\varphi)$, and $\mathrm{ds}_P(\varphi) = \mathrm{ds}(P_\varphi)$ if $\mathrm{ds}(P_\varphi)$ is defined.
If $P = \mathcal{T}_\Sigma$ then we omit P and write $\mathrm{RS}(\varphi)$, $\mathrm{ds}(\varphi)$ instead of $\mathrm{RS}_P(\varphi)$ and $\mathrm{ds}_P(\varphi)$, respectively.

Ranks and spectra for sentences and properties

Definition. For a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P \subseteq \mathcal{T}_\Sigma$ we say that φ is *P-totally transcendental* if $\mathrm{RS}_P(\varphi)$ is an ordinal.
A sentence φ is *co-(P)-totally transcendental* if $\neg\varphi$ is P-totally transcendental.
We omit P and say about totally transcendental and co-totally transcendental sentences if $P = \mathcal{T}_\Sigma$.

Theorem

For a language Σ there is a totally transcendental sentence $\varphi \in \mathrm{Sent}(\Sigma)$ iff Σ has finitely many predicate symbols.

Ranks and spectra for sentences and properties

Definition. For a language Σ, a property $P \subseteq \mathcal{T}_\Sigma$, an ordinal α and a natural number $n \geq 1$, a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is called (P, α, n)-*(co-)rich* if $\mathrm{RS}_P(\varphi) = \alpha$ and $\mathrm{ds}_P(\varphi) = n$ (respectively, $\mathrm{RS}_P(\neg\varphi) = \alpha$ and $\mathrm{ds}_P(\neg\varphi) = n$).
A sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is called (P, ∞)-*(co-)rich* if $\mathrm{RS}_P(\varphi) = \infty$ (respectively, $\mathrm{RS}_P(\neg\varphi) = \infty$).
If $P = \mathcal{T}_\Sigma$ we write that φ is (α, n)-(co-)rich instead of (P, α, n)-(co-)rich, and ∞-(co-)rich instead of (P, ∞)-(co-)rich.
If for a property P there is a $(P, *)$-(co-)rich sentence φ, we say that P has a $(P, *)$-*(co-)rich sentence*, where $* = \alpha, n$ or $\alpha = \infty$.

Ranks and spectra for sentences and properties

Theorem

(1) If a property $P \subseteq \mathcal{T}_\Sigma$ has a (P, α, m)-rich sentence φ which is (P, β, n)-co-rich then $\mathrm{RS}(P) = \max\{\alpha, \beta\}$, $\mathrm{ds}(P) = m$ for $\alpha > \beta$, $\mathrm{ds}(P) = n$ for $\alpha < \beta$, and $\mathrm{ds}(P) = m + n$ for $\alpha = \beta$.
(2) If for a property $P \subseteq \mathcal{T}_\Sigma$, $\mathrm{RS}(P) = \alpha$ and $\mathrm{ds}(P) = n$, then for each sentence $\varphi \in \mathrm{Sent}(\Sigma)$ the following assertions hold:
(i) $\mathrm{RS}_P(\varphi) \leq \alpha$,
(ii) if $\mathrm{RS}_P(\varphi) = \alpha$ then φ is (P, α, m)-rich for some $m \leq n$, and for $m = n$ either $\varphi \rhd_{\mathrm{tt}} P$ or φ is (P, β, k)-co-rich for some $\beta < \alpha$ and $k \in \omega$, and if $m < n$ then φ is $(P, \alpha, n - m)$-co-rich.

Ranks and spectra for sentences and properties

By Theorem 11 for any *e*-totally transcendental property P and any $\alpha \leq \mathrm{RS}(P)$ there are *s*-definable subfamilies P_φ with $\mathrm{RS}(P_\varphi) = \alpha$. Similarly all values $m \leq \mathrm{ds}(P)$ are also realized by appropriate *s*-definable subfamilies.
Thus the *spectrum* $\mathrm{Sp}_{\mathrm{Rd}}(P)$ for the pairs $(\mathrm{RS}_P(\varphi), \mathrm{ds}_P(\varphi))$ with nonempty P_φ forms the set

$$\{(\mathrm{RS}(P), m) \mid 1 \leq m \leq \mathrm{ds}(P)\} \cup \{(\alpha, m) \mid \alpha < \mathrm{RS}(P), m \in \omega \setminus \{0\}\},$$

which is an initial segment $O[(\beta, n)]$ consisting of all pairs $(\alpha, m) \in \mathrm{Ord} \times (\omega \setminus 0)$ with $\alpha \leq \beta$ and $m \leq n$ for $\alpha = \beta$, $\mathrm{RS}(P) = \beta$, $\mathrm{ds}(P) = n$.

Ranks and spectra for sentences and properties

Theorem

For any nonempty property $P \subseteq \mathcal{T}_\Sigma$ one of the following possibilities holds for some $\beta \in \mathrm{Ord}$ and $n \in \omega \setminus \{0\}$:
(1) $\mathrm{Sp}_{\mathrm{Rd}}(P) = O[(\beta, n)]$,
(2) $\mathrm{Sp}_{\mathrm{Rd}}(P) = \{\infty\}$,
(3) $\mathrm{Sp}_{\mathrm{Rd}}(P) = O[(\beta, n)] \cup \{\infty\}$.
All possibilities above are realized by appropriate languages Σ and properties $P \subseteq \mathcal{T}_\Sigma$.

Ranks and spectra for sentences and properties

The following theorem is shown in: Markhabatov N. D., Sudoplatov S. V. Ranks for families of all theories of given languages // Eurasian Mathematical Journal. — 2021. — Vol. 12, No. 2.

Theorem

Let $\mathcal{T}$ be a family of a countable language Σ and with $\mathrm{RS}(\mathcal{T}) = \infty$, α be a countable ordinal, $n \in \omega \setminus \{0\}$. Then there is a d_∞-definable subfamily $\mathcal{T}^* \subset \mathcal{T}$ such that $\mathrm{RS}(\mathcal{T}^*) = \alpha$ and $\mathrm{ds}(\mathcal{T}^*) = n$.

The latter two Theorems immediately imply:

Corollary

Let $\mathcal{T}$ be a family of a countable language Σ and with $\mathrm{RS}(\mathcal{T}) = \infty$, α be a countable ordinal, $n \in \omega \setminus \{0\}$. Then there is a d_∞-definable property $P \subset \mathcal{T}$ such that $\mathrm{Sp}_{\mathrm{Rd}}(P) = O[(\alpha, n)]$.

Links between sentences and properties

For a cardinality $\lambda \geq 1$, a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P \subseteq \mathcal{T}_\Sigma$ we write $\varphi \rhd^{\lambda}_{\mathrm{pt}} P$ if φ satisfies exactly λ theories in P, i.e., $|P_\varphi| = \lambda$.

By the definition if $P \neq \emptyset$ and $\varphi \rhd_{\mathrm{tt}} P$ then $\varphi \rhd^{|P|}_{\mathrm{pt}} P$, and conversely $\varphi \rhd^{|P|}_{\mathrm{pt}} P$ implies $\varphi \rhd_{\mathrm{tt}} P$ for finite P. For infinite P the converse implication can fail. Moreover, since infinite sets can be divided into two parts of same cardinality, one can easily introduce an expansion P' of P by a 0-ary predicate Q such that $Q \rhd^{|P'|}_{\mathrm{pt}} P'$ and $\neg Q \rhd^{|P'|}_{\mathrm{pt}} P'$, implying that $Q \not\rhd_{\mathrm{tt}} P'$.

Spectra for properties

For a property P we denote by $\mathrm{Sp}_{\mathrm{pt}}(P)$ the set $\{\lambda \mid \varphi \rhd^{\lambda}_{\mathrm{pt}} P$ for some sentence $\varphi\}$. This set is called the pt-*spectrum* of P.

Theorem

For any nonempty property $P \subseteq \mathcal{T}_\Sigma$ one of the following conditions holds:
(1) $\mathrm{Sp}_{\mathrm{pt}}(P) = (n+1) \setminus \{0\}$ for some $n \in \omega \setminus \{0\}$; it is satisfied iff P is finite with $|P| = n$;
(2) $\mathrm{Sp}_{\mathrm{pt}}(P) = Y \cup (n+1) \setminus \{0\}$ for some nonempty set $Y \subseteq |P|$ of infinite cardinalities and $n \in \omega \setminus \{0\}$;
(3) $\mathrm{Sp}_{\mathrm{pt}}(P) = Y \cup \omega \setminus \{0\}$ for some nonempty set $Y \subseteq |P|$ of infinite cardinalities;
(4) $\mathrm{Sp}_{\mathrm{pt}}(P) = Y$ for some nonempty set $Y \subseteq |P|$ of infinite cardinalities.
All values $(n+1) \setminus \{0\}$, $Y \cup (n+1) \setminus \{0\}$, $Y \cup \omega \setminus \{0\}$, and Y, for a nonempty set Y of infinite cardinalities and $n \in \omega \setminus \{0\}$, are realized as $\mathrm{Sp}_{\mathrm{pt}}(P)$ for an appropriate property P.

Spectra for properties

Theorem

For any nonempty E-closed property $P \subseteq \mathcal{T}_\Sigma$ with at most countable language Σ one of the following possibilities holds:
(1) $\mathrm{Sp}_{\mathrm{pt}}(P) = (n+1) \setminus \{0\}$ for some $n \in \omega \setminus \{0\}$, if P is finite with $|P| = n$;
(2) $\mathrm{Sp}_{\mathrm{pt}}(P) = \{2^\omega\} \cup (n+1) \setminus \{0\}$ for some $n \in \omega$, if P is infinite and has n isolated points;
(3) $\mathrm{Sp}_{\mathrm{pt}}(P) = (\omega+1) \setminus \{0\}$, if P is infinite and totally transcendental;
(4) $\mathrm{Sp}_{\mathrm{pt}}(P) = \{\omega, 2^\omega\} \cup \omega \setminus \{0\}$, if P has an infinite totally transcendental definable subfamily but P itself is not totally transcendental;
(5) $\mathrm{Sp}_{\mathrm{pt}}(P) = \{2^\omega\} \cup \omega \setminus \{0\}$, if P has infinitely many isolated points but does not have infinite totally transcendental definable subfamilies.

Definition. (Cf. [13]) For a property $P\subseteq\mathcal{T}_\Sigma$ a sentence $\varphi\in\mathrm{Sent}(\Sigma)$ is called *P-generic* if $\mathrm{RS}_P(\varphi)=\mathrm{RS}(P)$, and $\mathrm{ds}_P(\varphi)=\mathrm{ds}(P)$ if $\mathrm{ds}(P)$ is defined.
If $P=\mathcal{T}_\Sigma$ then we omit P and a P-generic sentence is called *generic*.

[13] Poizat, B. Groupes Stables. Nur Al-Mantiq Wal-Ma'rifah: Villeurbanne, France 1987. Truss, J.K. Generic Automorphisms of Homogeneous Structures // Proceedings of the London Mathematical Society. 1992, 65:3, 121–141. Tent, K., Ziegler, M. A Course in Model Theory // Lecture Notes in Logic. No. 40. Cambridge University Press: Cambridge, UK, 2012.

Proposition

Any P-generic sentence φ is $(P, \mathrm{RS}(P), \mathrm{ds}(P))$-rich if $\mathrm{RS}(P)$ is an ordinal, and (P, ∞)-rich if $\mathrm{RS}(P) = \infty$. And vice versa, each $(P, \mathrm{RS}(P), \mathrm{ds}(P))$-rich sentence, for an ordinal $\mathrm{RS}(P)$, is P-generic, and each (P, ∞)-rich sentence, for $\mathrm{RS}(P) = \infty$, is P-generic.

Corollary

If a property $P \subseteq \mathcal{T}_\Sigma$ is finite and $\varphi \in \mathrm{Sent}(\Sigma)$ then $\varphi \rhd_{\mathrm{tt}} P$ iff φ is P-generic.

Generic sentences

Proposition

For a property $P \subseteq \mathcal{T}_\Sigma$ there is a P-generic sentence $\varphi \in \mathrm{Sent}(\Sigma)$ with minimal/least P_φ iff P is finite. If that φ exists then $P_\varphi = P$.

Corollary

For any property $P \subseteq \mathcal{T}_\Sigma$ with $\mathrm{RS}(P) = \alpha \in \mathrm{Ord}$ and any sentence $\varphi \in \mathrm{Sent}(\Sigma)$ either φ is P-generic or $\neg\varphi$ is P-generic, or, for $\mathrm{ds}(P) > 1$ with non-P-generic φ and $\neg\varphi$, φ is represented as a disjunction of k $(P, \alpha, 1)$-rich sentences and $\neg\varphi$ is represented as a disjunction of m $(P, \alpha, 1)$-rich sentences such that $k + m = \mathrm{ds}(P)$, $k > 0$, $m > 0$.

Generic sentences

Theorem

(1) For any nonempty property $P \subseteq \mathcal{T}_\Sigma$ there are $\mathrm{ds}(P)$ P-generic theories if P is totally transcendental, and at least continuum many if P is not totally transcendental. In the latter case either all theories in P are P-generic if $\mathrm{Sp}_{\mathrm{Rd}}(P) = \{\infty\}$, or P has at least $\beta \cdot \omega + n$ non-P-generic theories if $\mathrm{Sp}_{\mathrm{Rd}}(P) = O[(\beta, n)] \cup \{\infty\}$.
(2) The CB-rank of each P-generic theory equals $\mathrm{RS}(P)$.

Generic sentences

Definition. For a property $P \subseteq \mathcal{T}_\Sigma$ a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is called *P-complete* if φ isolates a unique theory T in P, i.e., P_φ is a singleton. In such a case the theory $T \in P_\varphi$ is called *P-finitely axiomatizable* (by the sentence φ).

Proposition

For any nonempty property $P \subseteq \mathcal{T}_\Sigma$ a P-finitely axiomatizable theory T is P-generic iff P is finite.

Basedness of a theory

Definition [14]. A theory T is said to be Δ-*based*, where Δ is some set of formulas without parameters, if any formula of T is equivalent in T to a Boolean combination of formulas in Δ. For Δ-based theories T, it is also said that T has *quantifier elimination* or *quantifier reduction* up to Δ.

[14] Palyutin E. A., Saffe J., Starchenko S. S. Models of superstable Horn theories // Algebra and Logic. — 1985. — Vol. 24, No. 3. — P. 171–210.

Arities of formulae and theories

Definition. An elementary theory T is called *unary*, or 1-*ary*, if any T-formula $\varphi(\overline{x})$ is T-equivalent to a Boolean combination of T-formulas, each of which is of one free variable, and of formulas of form $x \approx y$.
For a natural number $n \geq 1$, a formula $\varphi(\overline{x})$ of a theory T is called *n-ary*, or an *n-formula*, if $\varphi(\overline{x})$ is T-equivalent to a Boolean combination of T-formulas, each of which is of n free variables.
For a natural number $n \geq 2$, an elementary theory T is called *n-ary*, or an *n-theory*, if any T-formula $\varphi(\overline{x})$ is n-ary.
A theory T is called *binary* if T is 2-ary, it is called *ternary* if T is 3-ary, etc.

Arities of formulae and theories

We will admit the case $n = 0$ for n-formulae $\varphi(\overline{x})$. In such a case $\varphi(\overline{x})$ is just T-equivalent to a sentence $\forall\overline{x}\varphi(\overline{x})$.
If T is a theory such that T is n-ary and not $(n-1)$-ary then the value n is called the arity of T and it is denoted by $\mathrm{ar}(T)$. If T does not have any arity we put $\mathrm{ar}(T) = \infty$.
Similarly, for a formula φ of a theory T we denote by $\mathrm{ar}_T(\varphi)$ the natural value n if φ is n-ary and not $(n-1)$-ary. If φ does not any arity we put $\mathrm{ar}_T(\varphi) = \infty$. If a theory T is fixed we write $\mathrm{ar}(\varphi)$ instead of $\mathrm{ar}_T(\varphi)$.

Arities of formulae and theories

By the definition any n-theory is Δ_n-based, where Δ_n consists of formulae with n free variables and formulae of the form $x \approx y$. It implies that theories of n-element models $\mathcal{M}$ are n-ary and based by formulae describing these n-element structures and differences/coincidences of elements.

Arities of formulae and theories

Proposition

A T-formula $\varphi(\overline{x})$ is not n-ary if and only if for any T-formulae $\psi_i(\overline{x}_i)$ with subtuples $\overline{x}_i$ of the tuple $\overline{x}$ having $l(\overline{x}_i) = n$ and $T \vdash \varphi(\overline{x}) \to \psi_i(\overline{x}_i)$, there exists a tuple $\overline{a} \in \mathcal{M} \models T$ such that $\mathcal{M} \models \psi_i(\overline{a}_i) \wedge \neg\varphi(\overline{a})$, where $\overline{a}_i$ is a subtuple of $\overline{a}$ consisting of substitutions of elements of $\overline{a}$ instead of correspondent elements of $\overline{x}_i$.

Proposition

A theory T of a language Σ is n-ary if and only if for any T-formula $\varphi(\overline{x})$ there is a finite sublanguage $\Sigma' \subseteq \Sigma$ such that $T \vdash \varphi(\overline{x}) \leftrightarrow \psi(\overline{x})$, where $\psi(\overline{x})$ is a Boolean combination of n-formulae.

Cartesian and mixed products, and sums

Since negations of formulas with n free variables again have n free variables, witnessing the n-arity of a formula it suffices to consider positive Boolean combinations of formulas with n free variables, i.e., conjunctions and disjunctions of formulas with n free variables. Thus for the description of definable sets for models $\mathcal{M}$ of n-theories it suffices describe links between definable sets A and B for n-formulas $\varphi(\overline{x})$ and $\psi(\overline{y})$, respectively, and definable sets C and D for $\varphi(\overline{x}) \wedge \psi(\overline{y})$ and $\varphi(\overline{x}) \vee \psi(\overline{y})$, respectively.

Cartesian and mixed products, and sums

If $\overline{x}=\overline{y}$ then $C=A\cap B$ and $D=A\cup B$, i.e., conjunctions and disjunctions work as set-theoretic intersections and unions.
If $\overline{x}$ and $\overline{y}$ are disjoint then $C=A\times B$ and $D=(A+B)_{\mathcal{M}}\rightleftharpoons \{\langle\overline{a},\overline{b}\rangle\mid \overline{a}\in A \text{ and } \overline{b}\in M, \text{ or } \overline{a}\in M \text{ and } \overline{b}\in B\}$, i.e., C is the Cartesian product of A and B, and D is the (*generalized*) *Cartesian sum* of A and B in the model $\mathcal{M}$.
If $\overline{x}\neq\overline{y}$, and $\overline{x}$ and $\overline{y}$ have common variables, then C and D are represented as a *mixed product* and a *mixed sum*, respectively, working partially as intersection and union, for common variables, and partially as Cartesian product and Cartesian sum, for disjoint variables.

Cylinders and projections

If $\overline{x}$ and $\overline{y}$ consist of pairwise disjoint variables and $\overline{x} \subsetneq \overline{y}$ then for any formula $\varphi(\overline{x})$ the set of solution of the formula $\varphi(\overline{x}) \wedge (\overline{y} \approx \overline{y})$ in $\mathcal{M}$ is called a *cylinder* with respect to $M^{l(\overline{y})}$ and generated by the set of solutions $\varphi(\mathcal{M})$. In any case generating sets for cylinders coincide their *projections*, i.e., sets of solutions for formulas $\exists\overline{z}\varphi(\overline{x})$, where $\overline{z} \subset \overline{x}$.
Since *n*-formulae produce cylinders on Cartesian products of universes, definable sets of *n*-ary theories are composed by Boolean combinations of definable cylinders, i.e., of elements of cylindric algebras.

Definition (cf. [15]). For a natural number n, a theory T is called n-transitive if each n-type $q(x_1, \ldots, x_n) \in S(T)$ is forced by its restriction to the empty language.

Proposition

If a theory T is n-transitive and non-$(n+1)$-transitive then T is not an n-theory.

[15] Sudoplatov S. V. Transitive arrangements of algebraic systems // Siberian Math. J. — 1999. — Vol. 40, No. 6. — P. 1142–1145.

Clearly, generic constructions [16] [17] allow to produce, for each $n \geq 1$, n-transitive and non-$(n+1)$-transitive theories with unique $(n+1)$-ary predicates and having quantifier elimination.

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[16] Sudoplatov S. V. Syntactic approach to constructions of generic models / S. V. Sudoplatov // Algebra and Logic. — 2007. — Vol. 46, No. 2. — P. 134–146.

[17] Sudoplatov S. V. Classification of Countable Models of Complete Theories. Novosibirsk : NSTU, 2018.

For instance, the theory T of structure $\mathcal{M} = \langle\{a, b, c, d\}; R^{(3)}\rangle$ with the ternary relation $R =$ $\{(a, b, c), (b, a, d), (b, c, d), (c, b, a), (a, c, d), (c, a, b), (c, d, a), (d, c, b), (d, a, b), (a, d, c), (b, d, a), (d, b, c)\}$ has quantifier elimination, is 2-transitive, not 3-transitive, and thus $\mathrm{ar}(T) = 3$. This example can be naturally spread for n-ary relations. In view of Proposition 3 it implies the following:

Corollary

For any natural $n \geq 1$ there is a theory T_n with $\mathrm{ar}(T_n) = n$.

The following examples illustrate values $\mathrm{ar}(T) = n$.

Example 1. [18] For any theory T_f of an unar, i.e., of one unary operation f, $\mathrm{ar}(T_f) \leq 2$. There are both theories T_{f_1} with $\mathrm{ar}(T_{f_1}) = 1$ and theories T_{f_2} with $\mathrm{ar}(T_{f_2}) = 2$. For instance, f_1 can be taken identical, and f_2 — a successor function on at least 3-element set.

Example 2. [14] For any theory T_Γ of an acyclic graph Γ with unary predicates, $\mathrm{ar}(T_\Gamma) \leq 2$.

[18] Sudoplatov S. V. Basedness of stable theories and properties of countable models with powerful types : Dis... cand. fiz.-mat. sc.: 01.01.06. — Novosibirsk, 1990. — 142 p. [in Russian]

Examples

Example 3. Let E be the following equivalence relation on the set $\mathbb{R}^n$:

$$\{(M,N)\mid M(x_1,\ldots,x_n),N(y_1,\ldots,y_n)\in\mathbb{R}^n, x_1^2+\ldots+x_n^2=y_1^2+\ldots+y_n^2\}.$$

Equivalence classes for the concentric spheres in $\mathbb{R}^n$ can not be reconstructed via cylinders defined by projections which form concentric balls and circles. The homogeneity of equivalence classes implies that each formula in the language $\langle E\rangle$ is reduced to a Boolean combination of $2n$-formulas. Thus $\mathrm{Th}(\langle\mathbb{R},E\rangle)$ is a $2n$-theory which is not an $(2n-1)$-theory.
Adding a disjoint unary predicate P and a bijection f between the set of spheres and P we obtain names for spheres and an additional coordinate for generating formulas for a basedness. Thus we form a $(2n+1)$-theory which is not an $2n$-theory.

Hence all possibilities for $\mathrm{ar}(T)=n$ are realized.

Example 4. Taking a non-degenerated algebraic surface at $\mathbb{R}^n$ which is not reduced to cylinders we obtain a defining formula $\varphi(\overline{x})$, $l(\overline{x}) = n + 1$, which is $(n + 1)$-formula and not an n-formula. In particular, non-degenerated non-cylindrical surfaces of the second order in $\mathbb{R}^3$ are defined by formulas φ with $\mathrm{ar}(\varphi) = 3$. For instance, taking the formula $x^2 + y^2 + z^2 = 1$ for the sphere S we obtain projections $x^2 + y^2 \leq 1$, $x^2 + z^2 \leq 1$, $y^2 + z^2 \leq 1$ which can not allow to reconstruct S by their Boolean combinations.

Example 5. Recall [19] [20] [21] that a *circular*, or *cyclic* order relation is described by a ternary relation K_3 satisfying the following conditions:
(co1) $\forall x \forall y \forall z (K_3(x,y,z) \rightarrow K_3(y,z,x))$;
(co2) $\forall x \forall y \forall z (K_3(x,y,z) \wedge K_3(y,x,z) \leftrightarrow x = y \vee y = z \vee z = x)$;
(co3) $\forall x \forall y \forall z (K_3(x,y,z) \rightarrow \forall t [K_3(x,y,t) \vee K_3(t,y,z)])$;
(co4) $\forall x \forall y \forall z (K_3(x,y,z) \vee K_3(y,x,z))$.
Clearly, $\mathrm{ar}(K_3(x,y,z)) = 3$ if the relation has at least three element domain. Hence, theories with infinite circular order relations are at least 3-ary.

[19] Kulpeshov B. Sh., Macpherson H.D. Minimality conditions on circularly ordered structures // Mathematical Logic Quarterly. — 2005. — Vol. 51, No. 4. — P. 377–399.

[20] Altaeva A. B., Kulpeshov B. Sh. On almost binary weakly circularly minimal structures // Bulletin of Karaganda University, Mathematics. — 2015. — Vol. 78, No. 2. — P. 74–82.

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Examples

The following generalization of circular order produces a *n-ball* or *n-circular* order relation, for $n \geq 4$, which is described by a n-ary relation K_n satisfying the following conditions:

(nbo1) $\forall x_1, \ldots, x_n (K_n(x_1, x_2, \ldots, x_n) \to K_n(x_2, \ldots, x_n, x_1))$;

(nbo2) $\forall x_1, \ldots, x_n \left(K_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) \wedge \right.$
$\left. K_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) \leftrightarrow \bigvee\limits_{i=1}^{n-1} x_i = x_{i+1} \right)$;

(nbo3) $\forall x_1, \ldots, x_n (K_n(x_1, \ldots, x_n) \to$
$\forall t [K_n(x_1, \ldots, x_{n-1}, t) \vee K_n(t, x_2, \ldots, x_n)])$;

(nbo4) $\forall x_1, \ldots, x_n (K_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) \vee$
$K_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n))$, $i < n$.

Clearly, $\mathrm{ar}(K_n(x_1, \ldots, x_n)) = n$ if the relation has at least n-element domain. Thus, theories with infinite n-ball order relations are at least n-ary.

Aritizable theories

Definition. A T-formula $\varphi(\overline{x})$ is called *n-expansible*, or *n-arizable*, or *n-aritizable*, if T has an expansion T' such that $\varphi(\overline{x})$ is T'-equivalent to a Boolean combination of T'-formulas with n free variables.
A theory T is called *n-expansible*, or *n-arizable*, or *n-aritizable*, if there is an n-ary expansion T' of T.
A theory T is called *arizable*, or *aritizable*, if T is n-aritizable for some n.
A 1-aritizable theory is called *unary-able*, or *unary-tizable*. A 2-aritizable theory is called *binary-tizable* or *binarizable*, a 3-aritizable theory is called *ternary-tizable* or *ternarizable*, etc.

Unary-tizable theories and formulae

Proposition

Any theory of a finite structure $\mathcal{M}$ is unary-tizable.

Proposition

Any formula of a theory having finitely many solutions is unary-tizable.

Theorem

A theory T is unary-tizable if and only if for any (some) model $\mathcal{M}$ of T any definable set is formed by unions, intersections, Cartesian sums and Cartesian products of subsets of M.

Binarizable theories and their definable sets

Similarly, all definable sets of binarizable theories are generated by unions, intersections, Cartesian sums and Cartesian products of subsets of M^2, extended by mixed sums and mixed products of these subsets and their combinations:

Theorem

A theory T is binarizable if and only if for any (some) model $\mathcal{M}$ of T any $\emptyset$-definable set is formed by unions, intersections, Cartesian sums, Cartesian products, mixed sums and mixed products of subsets of M^2.

n-aritizable theories and their definable sets

Theorem

A theory T is n-aritizable, for $n \geq 1$, if and only if for any (some) model $\mathcal{M}$ of T any $\emptyset$-definable set is formed by unions, intersections, Cartesian sums, Cartesian products, mixed sums and mixed products of subsets of M^n.

Theorem

A theory T is aritizable if and only if for any (some) model $\mathcal{M}$ of T any $\emptyset$-definable set is formed by unions, intersections, Cartesian sums, Cartesian products, mixed sums and mixed products of subsets of M^n, for some n.

Dynamics of arities

Proposition

A theory T has a non-aritizable expansion iff T has an infinite model.

Theorem

For any $\mu, \nu \in (\omega \setminus \{0\}) \cup \{\infty\}$ there is a theory $T_{\mu,\nu}$ and its expansion $T'_{\mu,\nu}$ such that $\mathrm{ar}(T_{\mu,\nu}) = \mu$ and $\mathrm{ar}(T'_{\mu,\nu}) = \nu$.

Theorem

If $\mathcal{T}\subseteq\mathcal{T}_\Sigma$, $T\in\mathrm{Cl}_E(\mathcal{T})\setminus\mathcal{T}$, $n\in\omega\setminus\{0\}$, then the theory T is n-ary iff any formula $\varphi(\overline{x})$ of T is T'-equivalent to some fixed Boolean combination of n-formulae for some infinite (e-minimal) set $\mathcal{T}'$ of theories T', which are obtained by restrictions of theories in $\mathcal{T}$ till the language $\Sigma(\varphi(\overline{x}))$, where $T_0\in\mathrm{Cl}_E(\mathcal{T}')$ for the $\Sigma(\varphi(\overline{x}))$-restriction T_0 of T.

A theory T satisfying the conditions of Theorem is called *uniformly n-approximated* by family $\mathcal{T}$.

Arities and E-closures

Corollary

Let $\mathcal{T} \subseteq \mathcal{T}_\Sigma$, $T \in \mathrm{Cl}_E(\mathcal{T}) \setminus \mathcal{T}$, $n \in \omega \setminus \{0\}$, and there are both an expansion $\mathcal{T}'$ of $\mathcal{T}$ and an expansion T' of T such that T' is uniformly n-approximated by $\mathcal{T}'$. Then T is n-aritizable (by the theory T').

Спасибо за внимание!